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SOME LINEAR PROGRAMMING MODELS FOR FORECASTING MANPOWER REQUIREMENTS OF NAVAL SHORE ACTIVITIES

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THESIS

SOME LINEAR PROGRAMMING MODELS
FOR FORECASTING MANPOWER REQUIREMENTS
OF NAVAL SHORE ACTIVITIES

by

Thomas Russell Sheridan

October 1969

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Some Linear Programming Models for Forecasting Manpower Requirements of Naval Shore Activities

by

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ABSTRACT

The problem of forecasting Naval shore station manning requirements is examined by considering the flow of goods and services. The approach of process analysis, which combines alternative productive processes, is used. Linear models are formulated for centralized planning at various management levels, from the operation of a single shore station to that of a command composed of several such stations. The application of the decomposition principle as a solution technique for centralized planning is further developed into a methodology for decentralized planning.

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I. PROBLEM OF MANPOWER FORECASTING

A. INTRODUCTION

An important problem of defense management is to determine force levels. The U. S. budget is the most stringent constraint on our defense posture, and hence it is desirable in the interests of the National security to efficiently utilize resources at all levels.

An important facet of the Navy's force level is represented by support activities ashore. Since these activities demand resources which would otherwise be allocated to combat forces, it is essential that such activities be operated efficiently.

The shore establishment may be considered to be similar to a business firm, because productivity is a measure of effectiveness of its operation. This is in contrast to combat forces where the measure used is qualitative, "combat effectiveness." Quantitive techniques will be developed to determine the most economical utilization of manpower resources within the bounds of specified productivity and personnel restraints.

B. PURPOSE AND MODEL REQUIREMENTS

The Navy's Shore Establishment is an extremely complex organization with many inter-connected components or subsystems. In order to be meaningful, any model of the Shore Establishment must represent these interrelations. It must impose such constraints on inputs and outputs as are applicable, and respond to changes in these values as they occur. Finally it must be compatible with the data that is, or can be, generated to measure those inputs and outputs.

The models developed here will be concerned only with those labor inputs that can be considered as variable. For instance, a station will have one and only one Commanding Officer in any circumstance, and thus no investigations will be made of this manpower category, and others which are similar.

C. SCOPE OF INVESTIGATION

This is an initial examination of manpower requirement forecasting. As such several model formulations will be examined. These address the problem of forecasting at different management levels and include centralized and decentralized approaches. Linear formulations are utilized in the models presented here, since even linear models may provide substantial insight into the interrelations of complex systems.

II. DEVELOPMENT OF THE GENERAL MATHEMATICAL MODEL

A. STRUCTURE OF THE MODEL

The models formulated for manpower forecasting are constructed by using process analysis. These models are linear programs which, when solved, yield the least-cost labor mix required to satisfy the capability requirements stipulated for that organization. Process analysis is an approach for analyzing production by considering alternative technological processes. Its application in this problem is detailed in Section B below.

For these models, the capability constraints are as follows:

1. Specified output levels must be maintained.
2. Consumption and Production of intermediate products must be related.
3. Upper and lower bounds on labor inputs are specified by availability of personnel, in the case of upper bounds. Lower bounds more commonly arise from the requirement that the Shore Establishment must provide sufficient billets that an acceptable sea-shore rotation can be maintained.
4. Policy constraints may arise from requirements for specified civilian to military personnel ratios at some stations, or in certain skill categories.
5. Variables must be non-negative.

B. PROCESS ANALYSIS

The manager of any organization faces the problem of finding the optimal utilization of such resources as are available in order to obtain required results. Often a choice must be made from among alternative productive processes capable of achieving the desired output. Thus, the manager is concerned not only with the allocation of resources,

but also with the selection of the techniques which will employ these resources. Process analysis models these technical alternatives by considering that the choice is to be made from a set of activities known as a technological matrix. Each activity in a process is made up of a combination of input variables in fixed quantitative ratio to produce certain output variables. Consideration of many activities permits the method to model the technical choices available to the manager. A simplified schematic representation of the application of process analysis is found in Appendix D.

C. MATHEMATICAL FORMULATION

The initial formulation is intended to model a single shore activity, for example, a Naval Air Station or Naval Shipyard. From this model extensions will be made to allow application at higher command levels.

1. Notation

Prior to formulation of the problem in mathematical terms it is necessary to define the following notation.

X - column vector whose elements (x_i) are the $i^{\underline{th}}$ skill category, in units of manhours per month.

Z - column vector whose elements (z_i) are the final products in the $i^{\underline{th}}$ category (e.g., trained pilots), in units of pilots per month.

Y - column vector whose elements (y_i) are the amount of the $i^{\underline{th}}$ intermediate product, (intermediate products are those which are internal to the station), in work units per month.

C - column vector of costs (c_i) of the $i^{\underline{th}}$ skill category input (x_i), in units of dollars per manhours per month.

W - column vector of activity levels (w_i) which represent the partial productivities of the technologies which make up the cost centers.

2. Initial Formulation

Process analysis is used in determination of the flow of inputs, intermediate products, and final outputs to and through the various subcost centers.

In mathematical form, the structure of the model set forth above becomes:

$$\text{minimize} \qquad C^T X$$

$$\text{subject to} \qquad Z \geq K_1$$

$$AW = \begin{bmatrix} Z \\ Y \\ X \end{bmatrix}$$

$$K_2 \leq X \leq K_3$$

$$W, X, Y, Z \geq 0$$

where C and X are (n_x X 1) column vectors, (C^T is the transpose of C), K_1 is a column vector of required output levels, K_2 and K_3 are column vectors of n_x lower and upper bounds on the labor category inputs, W is a column vector of m activity levels at which the cost centers will be operated, Z is a column vector of n_z final output categories, Y is a column vector of n_y intermediate products, and A is a matrix of technological coefficients (N X m) where $N = n_x + n_y + n_z$.

The formulation above has both X and W as unknown vectors. The formulation of a linear program in terms of the activity levels of the cost centers alone yields a solution.

3. Final Formulation as a Linear Program

In order to eliminate the vector of labor inputs and cast the problem in terms of activity levels as the only unknowns, the A matrix can be partitioned as follows:

$$A = \begin{matrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{matrix} \quad \text{, such that,} \quad \begin{matrix} A^{(1)} W = Z \\ A^{(2)} W = Y \\ A^{(3)} W = X \end{matrix}$$

It was originally assumed that intermediate products (Y) will sum out to zero for the station as a whole. This assumption will, however, lead to infeasibility, as was verified by empirical computation. The correct formulation of intermediate products is shown in Appendix B to be:

$$A^{(2)} W \geq 0$$

By use of the above partition of the matrix A, the final form of the linear program becomes one of finding the vector of optimal activity levels which will:

$$\begin{array}{ll} \text{minimize} & C^T A^{(3)} W \equiv \tilde{C}^T W \\ \text{subject to} & A^{(1)} W \geq K_1 \\ & A^{(2)} W \geq 0 \\ & K_2 \leq A^{(3)} W \leq K_3 \\ & W \geq 0 \; . \end{array}$$

4. Determination of Optimal Manning Requirements

The above linear program can be solved by any convenient L. P. algorithm to yield an optimal vector of activity levels, W*. The optimal manning level requirements X* can be determined from:

$$X^* = A^{(3)} W^* \; .$$

III. NATURE, SOURCES, AND AVAILABILITY OF DATA FOR MANPOWER ALLOCATION MODEL

In order that an operational model formulation be useful, it must have an empirical basis. In particular, the manpower allocation model delineated above requires empirical information related to the partial productivities of the alternative processes which reflect the various ways in which the cost center may be organized and/or operated. Hence it is appropriate to examine the management data reporting systems currently in use in the Navy to ensure that the model is designed to be compatible with available data.

A. RMS PRIME

The most promising data source, in terms of continuing availability, is the reporting systems RMS PRIME. This system has established a breakdown into cost centers and sub-cost centers, and specified work units to measure outputs. RMS PRIME reporting will eventually be extended to all stations. The reports provide the following data, at sub-cost center level, each month:

1. Civilian and military manhours and direct cost.
2. Total manhours.
3. Total costs for labor.
4. Total work units. (intermediate or final products)

Due to the fact that RMS PRIME is to be a Navy-wide reporting system, its use in the determination of required technological coefficients ensures adaptability of the model to any shore station where RMS PRIME data can be made available.

B. OTHERS

Another method which has been utilized to produce data required for the computation of technological coefficients is the industrial engineering technique of work sampling. This requires a work sampling team which observes personnel on the job and reports their activities in various productive (or non-productive) categories. The advantage of work sampling is that only actual productive time is accounted in that category, while in RMS reporting the tendency is to report all time spent on a job as productive time. The great expense of thorough work sampling, however, outweighs this advantage except perhaps as a spot-check to determine the accuracy of RMS data.

Another technique which has been found to yield results close to those of work sampling is that of questioning cost-center supervisors concerning labor inputs required to meet various output levels. While less costly than work sampling this method requires interviewers who are knowledgeable on the organization and operation of the cost centers involved.

IV. APPLICATION OF MANPOWER ALLOCATION MODEL AT VARIOUS LEVELS OF PLANNING

The mathematical model formulated in Chapter II may be applied at various command levels within the overall organization of the shore establishment. For illustration, the model will be applied to the Naval Air Basic Training Command, which consists of five training air stations, and nine squadrons. The organization of this command, and flow of student pilots through the various phases of training, are set forth in Appendix C.

The application of the allocation model to a single training air station, termed the Single Station Model (SSM), is to be investigated first.

A. SINGLE STATION MODEL

Using the notation defined in Chapter II, the final form of the linear program may be applied directly in the SSM. The linear programming problem which must be solved at each air station under the command of CNABATRA (Commander, Naval Air Basic Training) is the following:

$$\begin{array}{lrl} \text{minimize} & C^T A^{(3)} & W \\ \text{subject to} & A^{(1)} & W \geq K_1 \\ & A^{(2)} & W \geq 0 \\ & K_2 \leq A^{(3)} & W \leq K_3 \\ & & W \geq 0 \end{array}$$

where K_1 is the vector of specified levels of output which are to be maintained, and K_2 and K_3 are the vectors of lower and upper bounds on

labor inputs. As stated previously, by solution of the L.P. for an optimal vector of activity levels W* and the application of the relationship,

$$A^{(3)} W^* = X^*$$

the optimal manning level for the station in question can be determined.

A simplified example of a single station made up of three cost centers is presented in Appendix D.

B. CENTRALIZED PLANNING AT THE CNABATRA LEVEL

An extension of the Single Station Model above can be used to model centralized planning (CPM) for the entire CNABATRA organization. Consider the Naval Air Basic Training Command to be made up of s air stations. Due to the method of training pilots in phases, moving from one station to the next, some of the outputs (z_j) of the Single Station Model are now inputs to other stations within the organization. These products, e.g., partially trained pilots, are termed internal products. In order to maintain the distinction, the vector of internal products is designated T.

Double subscripting is utilized to distinguish between the inputs, outputs, etc., of the various stations. Thus the following notation arises:

X - column vector of labor inputs required by all stations.

X_j- column vector of labor inputs at the $j^{\underline{th}}$ station, such that,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_s \end{bmatrix}$$

X_{ij} - amount of the $i^{\underline{th}}$ labor input required at the $j^{\underline{th}}$ station.

Similarly for intermediate products (Y, Y_j, y_{ij}), final outputs (Z, Z_j, z_{ij}), cost center activity levels (W, W_j, W_{ij}), and labor input costs (C, C_j, c_{ij}). The column vector of internal products T, is defined such that t_{ij} is the $i^{\underline{th}}$ internal product produced at the $j^{\underline{th}}$ station. It is necessary to partition the $A^{(1)}$ matrix of technological coefficients for each of the stations to reflect the difference between final products and internal products. Designate the matrix formed by those rows of the $A^{(1)}$ matrix of the $j^{\underline{th}}$ station which correspond to internal products as $A_j^{(1)*}$ such that,

$$A_j^{(1)*} \; W_j = T_j$$

The matrix formed by the remaining rows (those relating final products at the $j^{\underline{th}}$ station), will be designated $A_j^{(1)}$. Note that the $A_j^{(1)}$ matrices of the Centralized Planning Model differ from those of the Single Station Models for each base by the deletion of the rows which make up $A_j^{(1)*}$.

Finally denote by $\bar{A}$ the matrix of technological coefficients for the entire organization. $\bar{A}$ is structured in the following manner:

$$\bar{A} = \begin{bmatrix} \bar{A}^{(1)} \\ \bar{A}^{(1)*} \\ \bar{A}^{(2)} \\ \bar{A}^{(3)} \end{bmatrix}$$

where:

$$\bar{A}^{(1)*} = \begin{bmatrix} A_1^{(1)*} & A_2^{(2)*} & \cdots & A_s^{(1)*} \end{bmatrix}$$

and

$$\bar{A}^{(k)} = \begin{bmatrix} A_1^{(k)} & 0 & \cdots & 0 \\ 0 & A_2^{(k)} & & \vdots \\ \cdot & \cdot & \cdots & 0 \\ 0 & \cdot & \cdots & A_s^{(k)} \end{bmatrix}$$

for $k = 1, 2, 3$.

An additional constraint set arises from the fact that there are upper (and possibly lower) bounds on the total number of personnel of each labor category which are available to the commander for assignment. The constraints take the mathematical form:

$$l_i \leq \sum_{j=1}^{s} x_{ij} \leq b_i \quad \text{for } i = 1,\ldots, n_X.$$

or, in the form required for inclusion in the linear program,

$$L \leq \sum_{j=1}^{s} A_j^{(3)} W_j \leq B$$

Thus, in the form previously standardized the problem facing the central planning commander becomes:

$$\begin{array}{ll} \text{minimize} & C^T\bar{A}^{(3)} W \\ \text{subject to} & \\ & \bar{A} W = \begin{bmatrix} Z \\ T \\ Y \\ X \end{bmatrix} \\ & Z \geq K \\ & L \leq \sum_{j=1}^{s} \bar{A}_j^{(3)} W_j \leq B \\ & W, X, Y, Z, T \geq 0. \end{array}$$

or

$$\begin{array}{ll} \text{minimize} & C^T\bar{A}^{(3)} W \\ & \bar{A}^{(1)} W \geq K \\ & \bar{A}^{(1)*} W \geq 0 \\ & \bar{A}^{(2)} W \geq 0 \\ & L \leq \sum_{j=1}^{s} \bar{A}^{(3)} W_j \leq B \\ & W_{ij} \geq 0, \text{ for } i = 1,\ldots, n_X;\ j = 1,\ldots, s. \end{array}$$

Note that the results of Appendix B concerning intermediate products have been extended to the formulation of internal products as well.

In this model, knowledge of technological coefficients of all air stations is required at CNABATRA level. In addition, CNABATRA, upon finding the optimal solution to the linear program, assigns personnel to the stations in his command <u>and</u> specifies the cost center in which they will be placed, as well as the activity level at which that cost center will be operated.

A simplified example of Centralized Planning applied to CNABATRA is presented in Appendix E.

V. COMPUTATIONAL ASPECTS

A. SOLUTION OF THE LINEAR PROGRAM

The extension of the Single Station Model formulation to centralized planning at the CNABATRA level produces a linear program which, though large, is within the capabilities of computer software packages such as MPS/360 for solution. For example, within the CNABATRA organization an average station could have approximately 200 labor inputs, 50 intermediate and final products, and 30 to 60 cost centers.

With the capability to handle 4095 constraint equations and unlimited variables, MPS/360 can be programmed for the centralized planning model of CNABATRA and its eight component stations. Note, however, that at command levels above that of CNABATRA a centralized planning model quickly becomes too large to be handled by any existing computer program.

B. DECOMPOSITION PRINCIPLE AS A SOLUTION TECHNIQUE

The principle of decomposition [DANZIG, 1963] was introduced as a solution technique for large linear programs representing complex super-systems in which the sub-systems operate almost independently but are tied together by some set of constraints on the super-system.

The general form of the constraint set of a problem readily amenable to the decomposition principle is as follows:

$$
\begin{array}{lllll}
A_1 X_1 & & & = & b_1 \\
& A_2 X_2 & & = & b_2 \\
& & \ddots & \vdots & \\
& & A_m X_m & = & b_m \\
A_1^* X_1 + & A_2^* X_2 \cdots & + A_m^* X_m & = & b_{m+1} \\
& & X_j \geq 0 & & \text{for all } j
\end{array}
$$

Where the A_j and the A_j^* are matrices, the X_j are vectors of activity levels, and the b_j are vectors of constants.

Inspection of the $\overline{A}$ matrix of the Centralized Planning Model indicates that the CPM may indeed be put in this form, that is:

$$\text{minimize} \qquad \sum_{j=1}^{S} C_j^T A_j^{(3)} W_j$$

$$\begin{array}{llr}
\text{subject to} \quad A_1 W_1 & = B_1 & (1) \\
\qquad A_2 W_2 & = B_2 & (2) \\
\qquad \ddots & & \\
\qquad A_s W_s & = B_s & (3) \\
A_1^{(1)*} W_1 + A_2^{(1)*} W_2 + \ldots + S^{(1)*} W_s & = T & (4) \\
L \leq A_1^{(3)} W_1 + A_2^{(3)} W_2 + \ldots + A_s^{(3)} W_s & \leq B & (5)
\end{array}$$

$$W_j \geq 0 \text{ for } j=1,\ldots,s$$

where

$$B_j = \begin{bmatrix} Z_j \\ Y_j \\ X_j \end{bmatrix}$$

Constraint sets (1), (2) and (3) are independent of each other, that is, the stations in CNABATRA are only jointly constrained by (4) and (5).

VI. DECENTRALIZED PLANNING AT CNABATRA LEVEL

The basic philosophy of decentralized planning is to enable a supersystem planner to optimize the operation of the entire organization under his cognizance without detailed information and central direction for each decision. The planner, who must satisfy overall constraints which bind together the entire organization, simply specifies the prices which subsystems must pay for scarce resources. By adjustment of those prices he "forces" the subsystems to seek resource requirement solutions which yield an optimal solution to the supersystem problem. Subsystem managers, who have the detailed knowledge of the productive processes necessary to determine optimal technological alternatives, are required to adjust their operations in the face of resource price changes in order to optimize the cost of operation at the new prices.

In this way, the supersystem planner is concerned only with the generation of resource requirements which are a feasible solution to supersystem constraints. Decisions concerning technological process and skill category mixers are delegated to the subsystem manager closest to those details.

The above considerations may be applied to obtain a decentralized planning model for the CNABATRA organization. That is, one in which the CNABATRA Planning Staff can be assured of reaching an optimal least-cost-labor solution to the Training Command, without being concerned with the organization or operation of any of the air stations which make up the Training Command.

It was shown in Chapter V that the only elements of the entire constraint set which jointly affect all the stations are:

$$A_1^{(1)*} W_1 + A_2^{(1)*} W_2 + \ldots + A_s^{(1)*} W_s = T \qquad (1)$$

and

$$L \leq A_1^{(3)} W_1 + A_2^{(3)} W_2 + \ldots + A_s^{(3)} W_3 \leq B \qquad (2)$$

Constraint set (1) above relates internal products. These internal products are final products in the Single Station Model and were redesignated as "internal" because they form inputs to stations at some advanced point in the training syllabus.

In the case of the Naval Air Basic Training Command these internal products are partially-trained student pilots advancing from one phase of training to the next. Thus the final output of completely trained pilots is the number of students who entered the program less the attrition at each phase of training. Sufficient data exist to determine the mean required output at each phase of the training syllabus in order that the final output of the system meet the specified level. In this manner output levels for each training station may be specified as elements of K_1, and the $A^{(1)*}$ constraint set returned to $A^{(1)}$. In this way, determination of the necessary levels of internal products allows them to be considered as final products again and constrained in the same manner as in the original Single Station Model. The station which received internal product (t_{ij}) will now consider that quantity as a labor input which has as a lower bound the level specified as an output requirement to the producing station. In this way constraint set (1) is eliminated as one held in common by all stations.

Constraint set (2) relates overall personnel availabilities and is as follows:

$$L \leq A_1^{(3)} W_1 + A_2^{(3)} W_2 + \ldots + A_s^{(3)} W_s \leq B$$

From Chapter II;

$$A_j^{(3)} W_j = X_j$$

for $j = 1, \ldots, s$.

which yields: $L \leq X_1 + X_2 + \ldots + X_s \leq B$

Note that in comparison with the general form of the decomposition algorithm the $\overline{A}_j$ are identity matrices in this formulation. (See Appendix F, equation (3)).

A supersystem linear program may now be written:

$$\text{minimize} \quad C^T X \tag{3}$$

$$\text{subject to} \quad L \leq X_1 + X_2 + \ldots + X_s \leq B \tag{4}$$

and further constrained that the X_j satisfy

$$\begin{array}{llll} A_1 W_1 & & & = B_1 \quad (5) \\ & A_2 W_2. & & = B_2 \quad (6) \\ & & \ddots & \vdots \\ & & A_s W_s & = B_s \quad (7) \end{array}$$

$W_j \geq 0$; for $j = 1, \ldots, s$.

where again,

$$B_j = \begin{bmatrix} Z_j \\ Y_j \\ X_j \end{bmatrix}$$

Thus, the decentralized Planner's concern is that the vector sum of the solutions X_j^*, submitted by the stations in his command, represent an optimal feasible solution to (3) and (4) above. X_j^* is determined by the j^{th} station by finding the optimal solution W_j^* to the subprogram:

minimize $C_j^T A_j^{(3)} W_j$

subject to
$$A_j^{(1)} W_j \geq K_{1j}$$
$$A_j^{(2)} W_j \geq 0$$
$$K_{2j} \leq A_j^{(3)} W_j \leq K_{3j}$$
$$W_j \geq 0$$

and solving for X_j^*, by use of $A_j^{(3)} W_j^* = X_j^*$

1. Decomposition Principle Applied to Decentralized Planning

As set forth in Appendix F, the Decentralized Planner "forces" the station to submit manpower requests which yield a feasible solution to the overall problem, by changing the prices to be charged for labor inputs. That is, in response to an increase in the price of labor inputs which violate the upper bound, stations will seek solutions which require less of that input. Similarly, a decrease in the price of a labor input will cause stations to attempt to utilize more of that input, with the result that the lower bound may be satisfied. It may be necessary that several iterations are required before feasibility of the supersystem program is reached.

2. Initiating the Algorithm

Assume that requests for manpower allocation $X_1^o, X_2^o, \ldots, X_s^o$, have been submitted by the stations as optimal solutions to their respective subprograms using the actual wages paid in the different labor categories. Further assume that these X_j^o do not form a feasible solution to the Master Program, i.e., some upper or lower bounds are violated. The phase I-phase II method of the Revised Simplex Method will, in general, be required to initiate the algorithm.

CNABATRA, as a Decentralized Planner, must change prices on labor inputs in such a way that, given the new prices, the stations will change their optimal labor mix to yield an optimal solution to the Master Program.

To determine the new set of prices, CNABATRA must solve this L.P., termed the restricted master program.

minimize

$$\sum_{j=1}^{s} C_j^T X_j^o \lambda_j$$

subject to

$$L \leq X_1^o \lambda_1 + X_2^o \lambda_2 + \dots + X_s^o \lambda_s \leq B \quad (n_x \text{ constraints}) \tag{8}$$

$$\begin{array}{llll} \lambda_1 & & = 1 & \\ & \lambda_2 & = 1 & s \text{ constraints} \\ & \ddots & \vdots & \\ & & \lambda_s = 1 & \end{array} \tag{9}$$

Let the lower bound L be equal to zero, for this example. The constraints on the subprogram ensure that $X_j = 0$. Note that the values of the λ_j are all one because there is only one basic feasible solution to the subprograms which is know.

Assuming that the vector X_j has n_x elements, the dual of this Master Program will have $n_x + s$ dual variables. The value of each of these dual variables can be thought of as the opportunity cost (or shadow price) of the input constraint with which it is associated.[2] Let the row vector of dual variables be designated:

$$V = \left[v_1, v_2, \dots, v_{n_x+s} \right]$$

---

[2]Hadley, G., Linear Programming, Addison-Wesley, p. 484, 1963.

In the standard Simplex minimization algorithm, a vector solution not presently in the basis will be considered for entry into the basis if its "$Z_j - C_j$" is positive, i.e., its indirect cost is greater than its direct cost. That vector X_j will be selected for entry which corresponds to the $\max_j (Z_j - C_j)$. In the restricted master program only those vectors which form the basis appear. Thus, the "$Z_j - C_j$" for vectors not in the basis cannot be calculated because those vectors have not even been determined. The decomposition principle provides a method for generation of new non-basic vectors <u>only</u> if they are likely candidates for entry into the basis.

The indirect cost, Z_j, is calculated in the Simplex method by the following equation:

$$Z_j = C_B^T B^{-1} a_j$$

where C_B^T is the transpose of the column vector of costs of variables in the basis, B^{-1} is the inverse of the present basis, and a_j is the column vector of coefficients of λ_j, j not in the basis.

From Duality: $V = C_B^T B^{-1}$

thus: $Z_j = V a_j$

The vector a_j of the Standard Simplex method corresponds in this case to; (refer to constraints (8) and (9))

$$a_j = \begin{bmatrix} X_j^o \\ e_s^j \end{bmatrix}$$

, where e_s^j is a s-dimensional column vector with a 1 in the $j^{\underline{th}}$ position and zeros elsewhere.

Then the Z_j for the $j^{\underline{th}}$ station is:

$$Z_j = v_1 x_{1j} + v_2 x_{2j} + \ldots + v_{n_x} x_{n_x j} + v_{n_x + j}$$

or:

$$Z_j - C_j = V X_j^o - C_j^T X_j^o + v_{n_x+j} = (V - C_j^T) X_j^o + v_{n_x+j}$$

The costs and the values of the dual variables are all known. A new vector to be brought into the basis must satisfy:

$$\max \ [(V - C_j^T) X + v_{n_x+j}] > 0$$

Note that v_{n_x+j} is a constant, thus the max $[(V - C_j^T) X_j]$ must be determined by the $j^{\underline{th}}$ subprogram to decide the vector which is to be entered into the basis. It is desirable to maintain the form of the subprogram as a minimization problem, in order to preserve the illusion (at the station level) that "prices" are simply being changed. This is accomplished by changing the objective function to the subprograms to:

$$\min \ [(C_j^T - V) X_j]$$

In terms of activity levels, the subprogram problem for the $j^{\underline{th}}$ station becomes:

$$\begin{aligned} \text{minimize} \quad & (C_j^T - V) A_j^{(3)} W_j \\ \text{subject to} \quad & A_j^{(1)} W_j \geq K_{1j} \\ & A_j^{(2)} W_j \geq 0 \\ K_{2j} \leq \ & A_j^{(3)} W_j \leq K_{3j} \\ & W_j \geq 0 \end{aligned}$$

After solution of the subprograms with the new prices, CNABATRA will receive new manning level requests, X_1^1 , X_2^1, ... , X_s^1, and corresponding costs of operating each station, these artificial "operating costs" are termed $\overline{C}_j$.

The negative of the $\overline{C}_j$ are added to v_{n_x+j}, and the X_j^1 corresponding to the max $[-C_j + v_{n_x+j}] > 0$ is selected for entry into the basis. Consider for example that:

$$\max\,[-C_j + v_{n_x+j}] = [-C_t + v_{n_x+t}]$$

Then X_t^1 will enter the basis of the restricted master program as the coefficient of a new variable λ_t^1. Thus the restricted master has the form:

minimize

$$C_1^T X_1^o \lambda_1 + C_2^T X_2^o \lambda_2 + \ldots + C_t^T\,(X_t^o \lambda_t + X_t^1 \lambda_t^1) + \ldots + C_s^T X_s^o \lambda_s$$

subject to

$$\begin{array}{llllll}
X_1^o\lambda_1 + X_2^o\lambda_2 + \ldots + X_t^o\lambda_t + X_t^1\lambda_t^1 + \ldots + X_s^o\lambda_s \le B & \\
\lambda_1 & = 1 \\
\quad \lambda_2 & = 1 \\
\qquad \ddots & \vdots \\
\qquad\quad \lambda_t + \lambda_t^1 & = 1 \\
\qquad\qquad \ddots & \vdots \\
\qquad\qquad\quad \lambda_s & = 1
\end{array} \qquad (10)$$

Note that at each iteration only one new vector of inputs and one corresponding variable, λ, is introduced into the restricted master.

3. Subsequent Iterations

Solution of the new restricted master leads to another vector of dual variables and computation of a new set of "prices". The prices are specified to the stations and a new determination of minimum operating is

made, allowing entry of a new vector solution into the basis of the master. This iterative process is continued until there is no j such that $-C_j + v_{n_x+j} > 0$. This condition ensures optimality in the master program.

In the general case where the lower bounds are greater than zero, the only modification to the above is that the constraint set (8) becomes twice as large.

That is:

$$L \leq X_1^o \lambda_1 + X_2^o \lambda_2 + \ldots + X_s^o \lambda_s \leq B$$

becomes:

$$X_1^o \lambda_1 + X_2^o \lambda_2 + \ldots + X_s^o \lambda_s \leq B$$

$$X_1^o \lambda_1 + X_2^o \lambda_2 + \ldots + X_s^o \lambda_s \geq L$$

Thus there $2n_x + s$ dual variables, the algorithm is operated as indicated in any case.

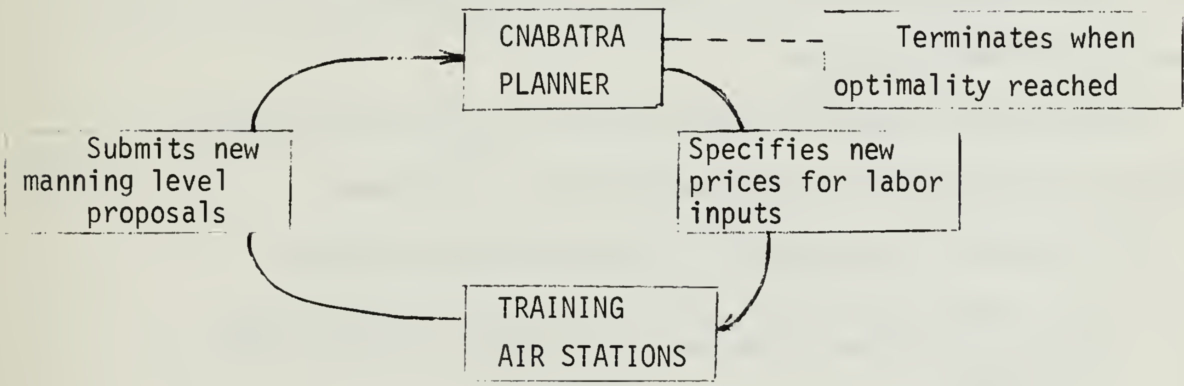
B. IMPLEMENTATION

The decentralized planning model could be implemented in the following manner at CNABATRA.

CNABATRA, having specified output levels which must be maintained by each of the stations in his command, directs that each station commander submit proposed manning levels which will produce the required outputs at a minimum expenditure for labor inputs, considering actual wage levels. At CNABATRA the proposals are combined and a restricted master program constructed using the proposed levels as the X_j vectors of the constraints. CNABATRA staff performs the computations to determine the necessary values which are specified to the stations as "new prices" of the labor inputs, and the stations are required to propose new manning

levels at the new prices. A new vector is selected for entry into the restricted master basis, a second set of artificial "prices" determined and new solutions requested from the station subprograms. This iterative procedure is continued until optimality is reached in the master program.

The flow of information can be represented as follows:



VII. OTHER PLANNING INFORMATION AVAILABLE FROM LINEAR PROGRAM SOLUTION

The utilization of the IBM linear programming package, MPS-360, greatly enhances the value of the models. Its particular advantage arises in the use of the Decentralized Planning Model. Upon solution of the restricted master, values of the dual variable activity can be putputted and a quick multiplication will yield the new objective function to be specified to the subprogram.

An advantage of MPS-360 is the ease with which post-optimality analysis of the solution thus generated can be performed.

The following investigations can be done by the system:

1. Over what range of values can the costs of labor inputs be varied without affecting the optimality of the present solution?

2. Over what range of values may the "right hand sides" be varied before feasibility of the present solution is affected? For instance, how low may the upper bound on personnel of a given labor category be set before a different solution is required?

3. Over what range may the values of the technological coefficients be varied without affecting the optimality of the present solution? This may be used to determine the impact of an anticipated change in technology of the present operations of an air station or a single cost center.

4. Use can be made of parametric programming to answer questions which are an extension of those posed above. For instance, if it is known that the required Pilot Training Rate can be expected to vary over a given range in subsequent time periods, parametric programming can be utilized to determine if the present solution will remain optimal over the entire range, or some portion of it, and can suggest a solution that will come "closest" in some sense, to an optimal long-run manning posture.

Both the post-optimality analysis and parametric programming techniques can be applied to as many individual values as desired, as well as to entire rows and/or columns, allowing the user an almost unlimited number of situations which can be simulated, and, in many cases, problems that can be foreseen and avoided.

VIII. CONCLUSIONS

The quantitative aspects of forecasting manpower requirements for a Naval Shore Activity have been examined and mathematical models have been formulated. Based on this study, the following conclusions have been reached:

1. The activities of the Naval shore establishment may be modelled using process analysis and linear programs developed to yield least-cost solutions for manning levels required to meet given output specifications.

2. In the centralized planning model, detailed information on each component activity must be available to the central planning staff. In contrast, for decentralized planning each subsystem manager is responsible for the efficient operation of activities under his control. The overall planner coordinates these activities by consideration of constraints known only at his level.

3. Decentralized planning by decomposition may be shown to reach the same optimal allocation solution as centralized planning.

4. The decentralized planning technique requires that the cost-center manager (who is closest to the actual operations) make organizational decisions utilizing his experience and encouraging his ingenuity.

5. Decentralized planning can be utilized at any management level, even at such low levels that no model is required for organization below.

APPENDIX A

<u>General Assumptions of Process Analysis Model</u>

A. Explicit Economic Assumptions of the Models are as follows:

1. Axiom of proportionality - requires that the partial productivities of all inputs are independent of activity levels. That is, the technological coefficients of the cost centers are not a function of the activity level at which the cost center is operated, or "constant returns to scale."

2. Axiom of additivity - requires that the combined output of two technological processes operated simultaneously is equal to the sum of the individual outputs when operated at the same activity levels that is, technological coefficients are not functions of interaction with other cost centers.

3. Maxim of economic efficiency - requires that whatever activity levels are set cost centers will each be operated as efficiently as possible.

4. Capital investment held constant during the period for which requirements are to be predicted. In the long run such changes in capital investment will, however be reflected in changes in technological coefficients to reflect increased (decreased) efficiency of operation.

APPENDIX B

FORMULATION OF INTERMEDIATE PRODUCT CONSTRAINTS

"A mathematical problem which is to correspond to physical reality should satisfy the following basic requirements:

1. The solution must exist.
2. The solution should be uniquely determined.
3. The solution should depend continuously on the data, (requirement of stability).

Any problem which satisfies our three requirements will be called a properly-posed problem."[1]

In order that the formulation of the allocation model as a linear program be "properly-posed," the requirements stated above must be restated as follows:

1. A feasible solution must exist.
2. While alternative optimal labor mixes may exist, the cost of the optimal solution will be unique.
3. At the least, piece-wise linearity of outputs as a function of inputs should be realized.

The assumption that intermediate products sum to zero has intuitive appeal, as the intermediate products generated as a positive output from one cost center will be inputs (negative) to another cost center. It will be shown that this formulation results in an infeasible problem.

---

[1]Courant, R. and Hilbert D., Methods of Mathematical Physics, Vol. II, p. 227, Interscience, 1962.

A. DISCUSSION

The L.P. model has been formulated as:

$$\text{minimize} \qquad C^T A^{(3)} W$$

$$\text{subject to} \qquad A^{(1)} W \geq K_1$$

$$A^{(2)} W = Y$$

$$K_2 \leq A^{(3)} W \leq K_3$$

$$W \geq 0$$

The usual assumption made is that the intermediate products sum to zero throughout the station, which would yield

$$A^{(2)} W = 0$$

Consider first the case of one technology per cost center. Thus if there are N cost centers, W is a N x 1 column vector.

Let:

f = number of final products

I = number of Intermediate Products

N = number of Cost Centers

Theorem 1: $I \geq N - f$

Proof: Each cost center has some type of unique quantifiable output, otherwise the elements of the A matrix for that cost center would be undetermined. The least number of intermediate products correspond to the case where each cost center produces one, and only one, output, (either final or intermediate). Thus the minimum I is equal to N - f. In general, I will be greater than N - f.

$A^{(2)}$ is an I x N matrix, since there is one row of $A^{(2)}$ corresponding to each intermediate product, and one column corresponding to each cost center

Thus the maximum rank of the $A^{(2)}$ matrix is:

$$r[A^{(2)}] = \min [I, N] \tag{1}$$

Theorem 2: Given that the $A^{(1)}$ matrix is of full rank, f; The rank of $A^{(2)}$ must be less than or equal to (N - f) for a feasible solution to exist.

Proof: By Theorem 1, $I \geq N - f$. Consider the following two cases:

Case 1: $I = N - f$

By (1) above, the maximum rank of $A^{(2)}$ is the min [I, N], and the theorem is satisfied.

Case 2: $I > N - f$

By contradiction, assume that $r(A^{(2)}) > N - f$ and a feasible solution exists. Consider the matrix:

$$A = \begin{bmatrix} A^{(1)} \\ \hline A^{(2)} \end{bmatrix}$$

The assumption above that the $r(A^{(2)}) = M > N - f$, implies f linearly independent rows of $A^{(1)}$ and M linearly independent rows of $A^{(2)}$. Denote the f rows from $A^{(1)}$ as $\{\beta_i^{(1)}\}$ and the M linearly independent rows of $A^{(2)}$ as $\{\beta_i^{(2)}\}$.

The members of the set $\{\beta_i^{(1)}\}$ cannot be linearly dependent on those of $\{\beta_i^{(2)}\}$. Otherwise that member could be written:

$$\beta_i^{(1)} = \sum_{j=1}^{m} \lambda_j \beta_j^{(2)}$$

And multiplying on the right by W:

$$\beta_i^{(1)} W = \sum_{j=1}^{m} \lambda_j \beta_j^{(2)} W = \lambda \beta^{(2)} W = \lambda \cdot 0 = 0$$

where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$

That is, $\beta_i^{(1)} W = 0$, which does not satisfy the constraints on final outputs. Thus there can be no linear dependencies between $\left\{ \beta_i^{(1)} \right\}$ and $\left\{ \beta_i^{(2)} \right\}$. Hence the rank of A is equal to $M + f > N - f + f = N$, and the contradiction is reached $r(A) > N$.

Thus, rank of $A^{(2)}$ must be less than or equal to $n - f$.

B. CONCLUSIONS

By Theorem 2, no feasible solution exists to the linear program with the constraint set $A^{(2)} W = 0$ unless the rank of $A^{(2)}$ is less than or equal to $N - f$. Since it is unlikely that actual linear dependencies between rows of the $A^{(2)}$ matrix would be detected due to the imprecise nature of the derivation of technological coefficients and computer round-off. In consideration of the above, the proper formulation of the linear program is as follows:

$$
\begin{aligned}
\text{minimize} \quad & C^T A^{(3)} W \\
\text{subject to} \quad & A^{(1)} W \geq K_1 \\
& A^{(2)} W \geq 0 \\
& K_2 \leq A^{(3)} W \leq K_3 \\
& W \geq 0.
\end{aligned}
$$

APPENDIX C

SYNOPSIS OF CNABATRA ORGANIZATION

The Naval Air Training Command consists of five airfields, with a total of nine training squadrons. The flow of pilots through the command is as follows. Primary flight training is conducted by VT - 1 at Saufley Field for all student pilots. Upon completion of the primary syllabus those students selected for jet training advance to Basic Jet Training at Meridian, Mississippi, (VT - 7 and VT - 9), then to Sherman Field (VT - 4) for aircraft carrier qualification and finally are sent to either Kingsville or Chase, Texas for advanced jet training.

Students selected for propellor aircraft training upon completion of the basic syllabus advance to Whiting Field, Milton, Florida for basic prop training (VT - 2 and VT - 3), then return to Saufley for carrier qualification (VT - 5). At this point another split is made, some students proceed to Corpus Christi, Texas for advanced training in multi-engined aircraft, while others report to Sherman Field (VT - 6) for advanced instruction before beginning heliocopter training at Ellyson Field (HT - 8).

APPENDIX D

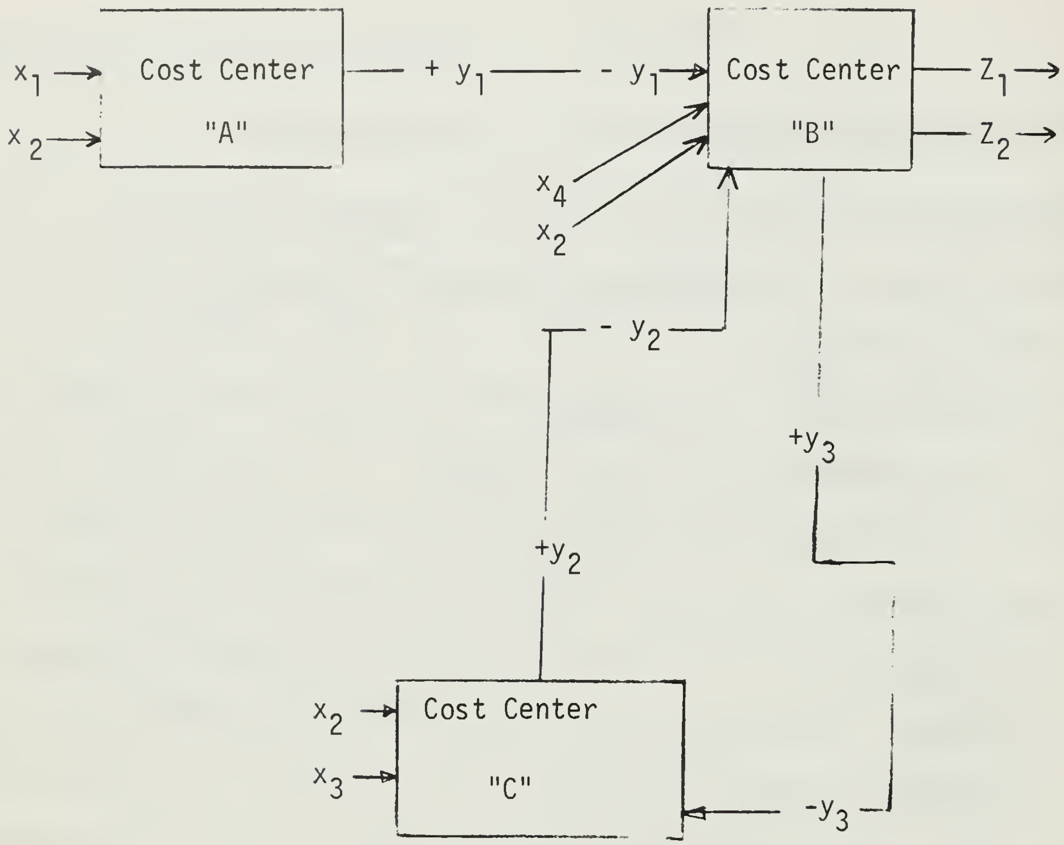
Simplified Example of Allocation Model for a Single Station

For purposes of illustration the allocation model set forth in Chapter IV is applied to a simplified air station consisting of:

1. Three cost centers:
    a. Administration - Cost Center A
    b. Aircraft Maintenance - Cost Center B
    c. Flight Training - Cost Center C
2. Four Variable Labor Inputs
    a. Students - x_1
    b. Airmen - E-2/E-3 - x_2
    c. Aviation mechanics - x_3
    d. Flight instructors - x_4
3. Three Intermediate Products
    a. Students processed - y_1
    b. Repaired aircraft - y_2
    c. Aircraft requiring repair - y_3
4. Two Final Outputs
    a. Student attrition - z_1
    b. Trained pilots - z_2

The values of the technological coefficients (a_{ij}, b_{ij}, and c_{ij} for the respective cost centers) are determined by the use of material balance equations relating inputs and outputs for each productive process in each cost center.

The following flow diagram represents the station thus generated:



For this example only one productive process considered in each cost center, yielding the following vector equations for each cost center:

Cost Center A:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \alpha_1 = \begin{bmatrix} y_1 \\ x_1 \\ x_2 \end{bmatrix}$$

Cost Center B:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ -b_{31} \\ -b_{41} \\ b_{51} \\ b_{61} \\ b_{71} \end{bmatrix} \beta_1 = \begin{bmatrix} z_1 \\ z_2 \\ y_1 \\ y_2 \\ y_3 \\ x_2 \\ x_4 \end{bmatrix} \tag{2}$$

Cost Center C:

$$\begin{bmatrix} c_{11} \\ -c_{21} \\ c_{31} \\ c_{41} \end{bmatrix} \gamma_1 = \begin{bmatrix} y_2 \\ y_3 \\ x_2 \\ x_3 \end{bmatrix} \tag{3}$$

where α_1, β_1, γ_1, are the activity levels at which the first productive process in each cost center is to be operated. As indicated the sign convention will be used that intermediate goods produced are positive, those consumed are negative.

Combining equations (1), (2), and (3) yields:

$$\begin{bmatrix} 0 & b_{11} & 0 \\ 0 & b_{21} & 0 \\ a_{11} & -b_{31} & 0 \\ 0 & -b_{41} & c_{11} \\ 0 & b_{51} & -c_{21} \\ a_{21} & 0 & 0 \\ a_{31} & b_{61} & c_{31} \\ 0 & 0 & c_{41} \\ 0 & b_{71} & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ y_1 \\ y_2 \\ y_3 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Note that the partition of the coefficient matrix indicated corresponds to:

$$\begin{bmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{bmatrix} W = \begin{bmatrix} Z \\ Y \\ X \end{bmatrix}$$

Assuming constraints on inputs and outputs are set as:

$$Z_2 \geq K_1$$

$$K_{11} \leq X_1 \leq K_{12}$$

$$K_{21} \leq X_2 \leq K_{22}$$

$$K_{31} \leq X_3 \leq K_{32}$$

$$K_{41} \leq X_4 \leq K_{42}$$

The problem can be written in the standard form as

$$\text{minimize} \quad (c_1 a_{21} + c_2 a_{31})\,\alpha_1 + (c_2 b_{61} + c_4 b_{71})\beta_1 + (c_2 c_{31} + c_3 c_{41})\gamma_1$$

subject to:

$$
\begin{array}{llllll}
b_{21}\ \beta_1 \geq K_1 & & & & & \\
 & a_{11}\,\alpha_1 & -b_{31}\,\beta_1 & & \geq & 0 \\
 & & -b_{41}\,\beta_1 & + C_{11}\,\gamma_1 & \geq & 0 \\
 & & b_{51}\,\beta_1 & - C_{21}\,\gamma_1 & \geq & 0 \\
k_{11} \leq & a_{21}\,\alpha_1 & & & \leq & k_{12} \\
k_{21} \leq & a_{31}\,\alpha_1 & +b_{61}\,\beta_1 & + C_{31}\,\gamma_1 & \leq & k_{22} \\
k_{31} \leq & & & C_{41}\,\gamma_1 & \leq & k_{32} \\
k_{41} \leq & & b_{71}\,\beta_1 & & \leq & k_{42}
\end{array}
$$

$$\alpha_1,\ \beta_1,\ \gamma_1,\ \geq\ 0$$

APPENDIX E

Simplified Model of Centralized Planning for CNABATRA

For illustration of the Centralized Planning Model (CPM) presented in Chapter VI, consider CNABATRA to be composed of two training fields, NAS-1 and NAS-2. Each of these will be considered to be structured much the same as the SSM examnle in Appendix B. That is, with but three cost centers: Administration, Aircraft Maintenance, and Flight Training.

Let the flow of products within and through the system be as shown in Figure E-1.

Cost Center A_j - Administration at station j.

Cost Center B_j - Flight training at station j.

Cost Center C_j - Aircraft maintenance at station j.

x_{1j} - student input to station j.

x_{2j} - E-2/E-3 input to station j.

x_{3j} - aviation mechanic input to station j.

x_{4j} - flight instructor input to station j.

y_{1j} - students processed by administration at station j.

y_{2j} - "down" aircraft at station j.

y_{3j} - repaired aircraft at station j.

z_{1j} - student attrition at station j.

z_{22} - qualified pilots (NAS-2 only)

t_{12} - partially trained pilots transferred from NAS-1 to NAS-2 for advanced training.

(Note that t_{12} is the same product as z_2 in the SSM, Appendix D).

Assuming, as before, one technology per cost center, vector equations for each of the stations are as follows.

For NAS-1:

Cost Center A_1:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} w_{11} = \begin{bmatrix} y_{11} \\ x_{11} \\ x_{21} \end{bmatrix}$$

Cost Center B_1:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \\ b_{51} \\ b_{61} \end{bmatrix} w_{12} = \begin{bmatrix} z_{11} \\ t_{12} \\ y_{11} \\ y_{21} \\ y_{31} \\ x_{41} \end{bmatrix}$$

Cost Center C_1:

$$\begin{bmatrix} -c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} w_{13} = \begin{bmatrix} y_{21} \\ y_{31} \\ y_{31} \end{bmatrix}$$

For NAS-2:

Cost Center A_2:

$$\begin{bmatrix} -d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} w_{21} = \begin{bmatrix} t_{12} \\ y_{12} \\ x_{22} \end{bmatrix}$$

Cost Center B_2:

$$\begin{bmatrix} e_{11} \\ e_{21} \\ -e_{31} \\ e_{41} \\ -e_{51} \\ e_{67} \end{bmatrix} w_{22} = \begin{bmatrix} z_{21} \\ z_{22} \\ y_{11} \\ y_{22} \\ y_{32} \\ x_{42} \end{bmatrix}$$

Cost Center C_2:

$$\begin{bmatrix} -f_{11} \\ f_{22} \\ f_{31} \end{bmatrix} w_{23} = \begin{bmatrix} y_{22} \\ y_{32} \\ x_{32} \end{bmatrix}$$

Equations can be grouped as before, yielding:

$$
\begin{array}{l}
\\
\bar{A}^{(1)} \\
\\
\bar{A}^{(1)*} \\
\\
\\
\\
\bar{A}^{(2)} \\
\\
\\
\\
\\
\\
\bar{A}^{(3)} \\
\\
\\
\\
\end{array}
\left[\begin{array}{cccccc}
0 & b_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e_{11} & 0 \\
0 & 0 & 0 & 0 & e_{21} & 0 \\
\hline
0 & b_{21} & 0 & -d_{11} & 0 & 0 \\
\hline
a_{11} & -b_{31} & 0 & 0 & 0 & 0 \\
0 & b_{41} & -c_{11} & 0 & 0 & 0 \\
0 & -b_{51} & c_{31} & 0 & 0 & 0 \\
0 & 0 & 0 & d_{21} & -e_{31} & 0 \\
0 & 0 & 0 & 0 & e_{41} & -f_{11} \\
0 & 0 & 0 & 0 & -e_{51} & f_{21} \\
\hline
a_{21} & 0 & 0 & 0 & 0 & 0 \\
a_{31} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{31} & 0 & 0 & 0 \\
0 & b_{61} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_{31} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_{31} \\
0 & 0 & 0 & 0 & e_{61} & 0
\end{array}\right]
\begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \\ w_{21} \\ w_{22} \\ w_{23} \end{bmatrix}
=
\left[\begin{array}{c}
z_{11} \\ z_{21} \\ z_{22} \\ \hline t_{12} \\ \hline y_{11} \\ y_{21} \\ y_{31} \\ y_{12} \\ y_{22} \\ y_{32} \\ \hline x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{22} \\ x_{32} \\ x_{42}
\end{array}\right]
$$

$$
\sum_{j=1}^{2} A_j^{(3)} w_j \qquad
\begin{bmatrix}
a_{21} & 0 & 0 & 0 & 0 & 0 \\
a_{31} & 0 & 0 & d_{31} & 0 & 0 \\
0 & 0 & c_{31} & 0 & 0 & f_{31} \\
0 & b_{61} & 0 & 0 & e_{61} & 0
\end{bmatrix}
\begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \\ w_{21} \\ w_{22} \\ w_{23} \end{bmatrix}
=
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

The constraints imposed by $A_1^{(1)}$ are independent of those imposed by $A_2^{(1)}$, and similarly for the $A_j^{(2)}$ and $A_j^{(3)}$, $j = 1,2$.

Assuming the following bounds have been specified:

$$z_{22} \geq K_1$$

$$\alpha_{ij} \leq x_{ij} \leq \beta_{ij} \qquad \begin{array}{l} i = 1,\ldots,4 \\ j = 1,2 \end{array}$$

and,

$$l_i \qquad \sum_{j=1}^{2} x_{ij} \leq b_i \qquad i = 1,\ldots,4$$

The problem is specified by:

minimize $(C_{11}a_{21} + C_{21}a_{31})\, w_{11} + (C_{41}b_{61})\, w_{12} +$

$(C_{31}c_{31})\, w_{13} + (C_{22}d_{31})\, w_{21} + (C_{42}e_{61})\, w_{22} +$

$(C_{32}f_{31})\, w_{23}$

subject to:

$$\begin{array}{lllllllll}
 & & & & & e_{21}w_{22} & & \geq & K_1 \\
 & & b_{21}w_{12} & & -d_{11}w_{21} & & & \geq & 0 \\
 & a_{11}w_{11} & -b_{31}w_{12} & & & & & \geq & 0 \\
 & & b_{41}w_{12} & -c_{11}w_{13} & & & & \geq & 0 \\
 & & -b_{51}w_{12} & +c_{31}w_{13} & & & & \geq & 0 \\
 & & & & d_{21}w_{21} & -e_{31}w_{22} & & \geq & 0 \\
 & & & & & e_{41}w_{22} & -f_{11}w_{23} & \geq & 0 \\
 & & & & & -e_{51}w_{22} & +f_{21}w_{23} & \geq & 0 \\
\alpha_{11} \leq & a_{21}w_{11} & & & & & & \leq & \beta_{11} \\
\alpha_{21} \leq & a_{31}w_{11} & & & & & & \leq & \beta_{21} \\
\alpha_{31} \leq & & & c_{31}w_{13} & & & & \leq & \beta_{31} \\
\alpha_{41} \leq & & b_{61}w_{12} & & & & & \leq & \beta_{41} \\
\alpha_{22} \leq & & & & d_{31}w_{21} & & & \leq & \beta_{22} \\
\alpha_{32} \leq & & & & & & f_{31}w_{23} & \leq & \beta_{32} \\
\alpha_{42} \leq & & & & & e_{61}w_{22} & & \leq & \beta_{42}
\end{array}$$

$$
\begin{array}{llllll}
l_1 & \leq a_{21}w_{11} & & & & \leq b_1 \\
l_2 & \leq a_{31}w_{11} & & & +d_{31}w_{21} & \leq b_2 \\
l_3 & \leq & & c_{31}w_{12} & & f_{31}w_{23} \leq b_3 \\
l_4 & \leq & b_{61}w_{12} & & e_{61}w_{22} & \leq b_4
\end{array}
$$

and:

$$w_{ij} = 0, \text{ for all i and j}$$

Solution of the L.P. will yield an optimal set of activity levels:

$$
w^* = \begin{bmatrix} w_{11}^* \\ w_{12}^* \\ w_{13}^* \\ w_{21}^* \\ w_{22}^* \\ w_{23}^* \end{bmatrix}
$$

The optimal manning levels can be determined by

$$
\begin{aligned}
x_{11}^* &= a_{21}w_{11}^* \\
x_{21}^* &= a_{31}w_{11}^* \\
x_{31}^* &= c_{31}w_{31} \\
x_{41}^* &= b_{61}w_{12}^* \\
x_{22}^* &= d_{31}w_{21}^* \\
x_{23}^* &= f_{31}w_{21}^* \\
x_{24}^* &= e_{61}w_{23}^*
\end{aligned}
$$

APPENDIX F

DECOMPOSITION PRINCIPLE

The decomposition principle is a technique which aids in the optimization of a complex supersystem composed of many subsystems which operate almost independently but have a few constraints and an objective function which are held in common. The supersystem is decomposed into subprograms which correspond to each of the independent parts, and a master program which ties the subprogram together.

The technique has the disadvantage that the master and subprograms may require solution several times.

The general form of decomposition will be discussed.

A. GENERAL FORM

Consider a supersystem made up of two such subsystems. The linear program for the supersystem may be formulated in the following manner:

$$\text{minimize} \quad C_1^T X + C_2^T Y$$

$$\text{subject to} \quad A_1 X = b_1 \qquad (1)$$

$$A_2 Y = b_2 \qquad (2)$$

$$\bar{A}_1 X + \bar{A}_2 Y = \bar{b} \qquad (3)$$

Where X and Y are vectors of activity levels in the first and second subsystem respectively. Equation (1) expresses the constraints which involve only the first subsystem, equation (2) is the constraints only on the second, and equation (3) expresses those constraints held in common.

Let S_1 and S_2 be the set of extreme point solutions to $A_1 X = b_1$ and $A_2 Y = b_2$ respectively. It is assumed that the elements of S_1 and S_2 form closed bounded sets. Under this assumption, any X a feasible solution of $A_1 X = b_1$ can be represented as a convex linear combination of the elements of S_1. Consider the elements of S_1 to be:

$$S_1 = X_1, X_2, \ldots, X_k$$

Then any solution X can be written:

$$X = \sum_{j=1}^{K} \lambda_j X_j$$

where

$$\sum_{j=1}^{K} \lambda_j = 1, \quad \lambda_j \geq 0$$

Similarly any feasible solution Y to $A_2 Y = b_2$ can be represented by a convex combination of the elements of S_2. That is:

$$Y = \sum_{j=1}^{L} \mu_j Y_j$$

$$\sum_{j=1}^{L} \mu_j = 1, \quad \mu_j \geq 0$$

where,

$$S_2 = \left\{ Y_1, Y_2, \ldots, Y_L \right\}$$

Thus for any feasible solutions X and Y, the supersystem program can be re-written in terms of the λ_j and μ_j, as follows:

$$\text{minimize} \quad C_1^T \left(\sum_{j=1}^{K} \lambda_j X_j \right) + C_2^T \left(\sum_{j=1}^{K} \mu_j Y_j \right)$$

$$\text{subject to} \quad \bar{A}_1 \left(\sum_{j=1}^{K} \lambda_j X_j \right) + \bar{A}_2 \left(\sum_{j=1}^{L} \mu_j Y_j \right) = \bar{b}$$

$$\sum_{j=1}^{K} \lambda_j \qquad\qquad\qquad = 1$$

$$\sum_{j=1}^{L} \mu_j = 1$$

$$\lambda_j, \mu_j \geq 0$$

Conversely any λ_j and μ_j satisfying the above determine solutions X and Y which are feasible solutions to the supersystem program.

Define the vectors:

$$G_1 = \begin{bmatrix} C_1^T X_1 \\ C_1^T X_2 \\ \cdot \\ \cdot \\ \cdot \\ C_1^T X_K \end{bmatrix}, \quad G_2 = \begin{bmatrix} C_2^T Y_1 \\ C_2^T Y_2 \\ \cdot \\ \cdot \\ \cdot \\ C_2^T L \end{bmatrix}$$

and the matrices:

$$F_1 = [\bar{A}_1 X_1, \bar{A}_1 X_2, \ldots, \bar{A}_1 X_K]$$

$$F_2 = [\bar{A}_2 Y_1, \bar{A}_2 Y_2, \ldots, \bar{A}_2 Y_L]$$

Designate the columns of F_1 as $(f_{11}, f_{12}, \ldots, f_{1K})$, similarly for F_2.

Then the supersystem program becomes:

minimize $\quad G_1 \lambda + G_2 \mu$

subject to $\quad F_1 \lambda + F_2 \mu = \bar{b}$

$$\sum_{j=1}^{K} \lambda_j = 1$$

$$\sum_{j=1}^{L} \mu_j = 1$$

$$\lambda_j, \mu_j \geq 0$$

The problem thus formulated, with all extreme point solutions to the systems $A_1 X = b_1$ and $A_2 Y = b_2$ known, is termed the Extremal or Master program. It has the drawback that the set of all extreme point solutions will not, in general, be known and will be difficult, (and unnecessary) to obtain.

Assume that some basic feasible solution to the master program is known. A new vector would be considered for entry if its "$Z_j - C_j$" is greater than zero. In the Simplex L.P. Algorithm, $Z_j - C_j$ is calculated from the formula $C_B^T B^{-1} a_j - C_j$, where C_B^T is the transpose of the costs of variables in the basis, B^{-1} is the universe of the present basis, and a_j is the vector of coefficients from the A matrix. From Duality, $C_B^T B^{-1}$ is equal to the vector of dual variables, V.[3]

Consider that the right hand side $\overline{b}$, represents m constraints. Then the dual of the master program will have m + 2 variables. The vector a_j can be represented as:

$$a_j = \begin{bmatrix} f_{1j} \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} f_{2j} \\ 0 \\ 1 \end{bmatrix}$$

Thus for vectors from Subprogram 1:

$$Z_j - C_j = Vf_{1j} - C_j + V_{m+1}$$

For vectors from Subprogram 2:

$$Z_j - C_j = Vf_{2j} - C_j + v_{m+2}$$

[3]Hadley, G., Linear Programming, p. 229, Addison-Wesley, 1963.

Recall:

$$f_{1j} = \bar{A}_1 X_j$$

$$f_{2j} = \bar{A}_2 Y_j$$

In order to choose a new vector to enter the basis, the subsystems are called upon to find another extreme point solution to each subprogram. Each subprogram determines in effect, which of its solutions yields a maximum "$Z_j - C_j$." This is brought about by the master programmer specifying new "prices" for the X's and Y's. The new problems specified to the subsystems are the following:

Subprogram 1:

$$\begin{aligned} &\text{maximize} && (V\bar{A}_1 - C_1^T)\, X \\ &\text{subject to} && A_1 X = b_1 \\ & && X \geq 0 \end{aligned} \tag{4}$$

Subprogram 2:

$$\begin{aligned} &\text{maximize} && (V\bar{A}_2 - C_2^T)\, Y \\ &\text{subject to} && A_2 Y = b_2 \\ & && Y \geq 0 \end{aligned} \tag{5}$$

Let the solutions to (4) and (5) be $\bar{C}_1$ and $\bar{C}_2$. The master programmer selects:

$$\min\, [\bar{C}_1 + v_{m+1}, \bar{C}_2 + v_{m+2}] > 0 \tag{6}$$

and enters the λ_{K+1} or μ_{L+1} which corresponds to (6) into the basis. Consider, for example, that λ_{K+1} will be brought into the basis. The new master problem becomes:

$$\begin{array}{lll} \text{minimize} & G_1\lambda + C_1^T X_{K+1} \lambda_{K+1} + G_2\mu & \\ \text{subject to} & F_1\lambda + \overline{A}_1 X_{K+1} \lambda_{K+1} + F_2\mu & = \overline{b} \\ & \sum_{j=1}^{K} \lambda_j + \lambda_{K+1} & = 1 \\ & \sum_{j=1}^{L} \mu_j & = 1 \\ & \lambda_j,\ \mu_j & = 0 \end{array}$$

Optimality is obtained if at some step:

$$[\overline{C}_1 + v_{m+1}] < 0$$

and

$$[\overline{C}_2 + v_{m+1}] < 0$$

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13. ABSTRACT

The problem of forecasting Naval shore station manning requirements is examined by considering the flow of goods and services. The approach of process analysis, which combines alternative productive processes, is used. Linear models are formulated for centralized planning at various management levels, from the operation of a single shore station to that of a command composed of several such stations. The application of the decomposition principle as a solution technique for centralized planning is further developed into a methodology for decentralized planning.

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